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AXISYMMETRIC, LAMINAR, BINARY SHEAR LAYER WITH THERMAL DIFFUSION AND DIFFUSION-THERMO EFFECTS*

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Abstract—Consideration is given to the thermal separation of components across the shear layer between a cold jet mixture (of helium and nitrogen) and a hot ambient gas, both initially at the same concentration levels. An asymptotic steady state solution is obtained assuming the layer to be thin relative to the jet radius. As a consequence of small inhomogeneity the momentum equation may be decoupled from energy considerations and *a priori* known velocity profiles result with the customary assumption of constant product of density and viscosity. Iterative integration of the concentration and energy relations is completed on the basis of the steady state condition of zero mass transport across the dividing streamline using similarity co-ordinates. Reasonable agreement with available experimental data results from use of the modified hard sphere model for the thermal diffusion ratio.

NOMENCLATURE

C_i,	mass concentration, ith component;
C_p,	specific heat, constant pressure;
$\tilde{c}$,	defined by equation (7);
D_{ij},	coefficient of mass diffusion;
f,	modified stream function;
h,	enthalpy;
J_m,	mass flux;
J_q,	energy flux;
k,	coefficient of thermal conductivity;
k_T,	thermal diffusion ratio;
$\tilde{k}_T$,	$k_T/(k_T)_{\text{rigid sphere}}$;
M,	Mach number;
m_{ij},	m_i/m_j;
m,	equivalent molecular weight of mixture, $m_1 m_2/[m_1 + (m_2 - m_1)C_1]$;
m_i,	molecular weight, ith component;
P,	pressure;
$\underline{P}$,	pressure tensor;
Pr,	Prandtl number, $C_p\mu/k$;
$\bar{q}$,	velocity vector;
R,	gas constant;
Sc,	Schmidt number, $\mu/\rho D_{ij}$;
t,	time;
T,	absolute temperature;
u, v,	velocity components along (x, y) directions;
x, y,	Cartesian coordinates;
α,	thermal diffusion factor;
β,	parameter in asymptotic momentum solution, equation (24);
γ,	ratio of specific heats;
η,	similarity parameter, equation (5);
λ,	mixture parameter;
μ,	coefficient of viscosity;
ρ,	density;
Φ,	dissipation function;
ψ,	stream function;
$(\,)_a$,	ambient;
$(\,)_j$	jet;
$(\,)_1$,	helium;
$(\,)_2$,	nitrogen;
$(\,)_0$,	average value;
$(\,)^*$,	quantities evaluated on dividing streamline;
$\overline{(\,)}$,	$(\,)/(\,)_a$.

INTRODUCTION

THE PRESENCE of a thermal diffusion effect in the classical case of dynamical equilibrium is well-known both theoretically and experimentally [1, 2]. Simple kinetic theory with a modification of k_T based on the rigid sphere model gives reasonable results. Thomann and Baron [3] and

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subsequently Finkleman [4] constructed a device to effectively determine thermal diffusion ratios under dynamical nonequilibrium conditions. They used a He–N_2 mixture initially at the same concentration levels both in a jet and in a stagnant ambient gas of finite volume (Fig. 1). The low speed jet was maintained at temperature levels near 70°K, while the ambient gas was at approximately room temperature, resulting in a temperature ratio across the interface of about 4. Thermally induced separation was determined from concentration level measurements both within the jet core and exterior to the interface mixing layer. An approximate analysis [3] involving constant property assumptions, $Le = 1$, $Pr = 1$ or ∞, indicated fairly good agreement with the data. However the crude assumptions employed there suggested that an exact calculation be attempted. The present work shows the approach and results for such an evaluation.

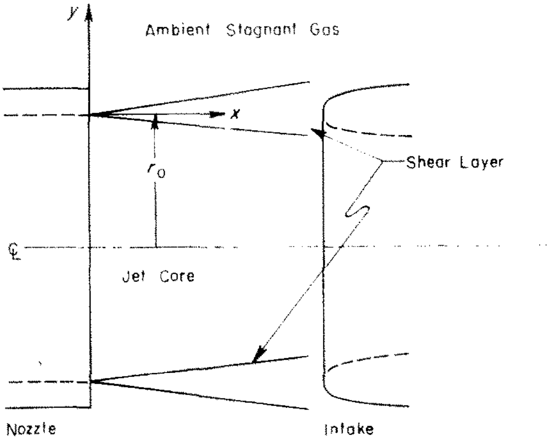

FIG. 1. Schematic of flow.

FORMULATION OF DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The governing differential equations for a binary mixture are [5]:

Species $$\rho \frac{DC_1}{Dt} \equiv \rho \left\{ \frac{\partial C_1}{\partial t} + \bar{q} \cdot \nabla C_1 \right\} = - \nabla \cdot J_{m_1}$$

Continuity $$\frac{D\rho}{Dt} = - \rho \nabla \cdot \bar{q}$$

Momentum $$\rho \frac{D\bar{q}}{Dt} = - \nabla \cdot \underline{P}$$

Energy $$\rho \frac{Dh}{Dt} = \frac{DP}{Dt} + \Phi - \nabla \cdot \bar{J}_q$$

where P, $\underline{P}$, and Φ are pressure, the pressure tensor and dissipation function, respectively. Neglecting pressure diffusion and external force field effects, the mass and energy flux vectors are

$$\bar{J}_{m_1} = - \rho D_{12} \left[\nabla C_1 + \frac{m_1 m_2}{m^2} k_T \nabla \ln T \right]$$

$$\bar{J}_q = - k \nabla T + \left[(h_1 - h_2) + RT \frac{k_T}{C_1 C_2} \right] \bar{J}_{m_1}.$$

The equations are coupled and nonlinear, and in view of the axisymmetric geometry the assumption is made that all transport processes occur only within a very thin layer compared with the radius of the jet. The relations then reduce to the equivalent two-dimensional problem (with co-ordinates as shown in Fig. 1):

$$\rho \left[u \frac{\partial C_1}{\partial x} + v \frac{\partial C_1}{\partial y} \right] = \frac{\partial}{\partial y} \left[\rho D_{12} \left(\frac{\partial C_1}{\partial y} + \frac{m_1 m_2}{m^2} k_T \frac{\partial \ln T}{\partial y} \right) \right] \tag{1}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{2}$$

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \tag{3}$$

$$\left. \begin{aligned} &\rho C_p \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = \mu \left(\frac{\partial u}{\partial y} \right)^2 \\ &\quad + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{RT k_T}{C_1 C_2} \\ &\left\{ \frac{\partial}{\partial y} \left[\rho D_{12} \left(\frac{\partial C_1}{\partial y} + \frac{m_1 m_2}{m^2} k_T \frac{\partial \ln T}{\partial y} \right) \right] \right\} \\ &\quad + \rho D_{12} \left\{ \frac{\partial C_1}{\partial y} + \frac{m_1 m_2}{m^2} k_T \frac{\partial \ln T}{\partial y} \right\} \\ &\quad \frac{\partial}{\partial y} \left\{ (h_1 - h_2) + \frac{RT k_T}{C_1 C_2} \right\} \end{aligned} \right\} \tag{4}$$

wherein the usual boundary layer assumptions have been made, and axisymmetric terms have been neglected as higher order contributions for a thin shear layer. Let

$$u = \frac{\rho_a}{\rho}\frac{\partial\psi}{\partial y}, \qquad v = -\frac{\rho_a}{\rho}\frac{\partial\psi}{\partial x}$$

and introduce similarity with the parameter

$$\eta = \tfrac{1}{2}\sqrt{\left(\frac{u_j}{\nu_a\, x\, \tilde{c}}\right)}\int_0^y \frac{\rho}{\rho_a}\,\mathrm{d}y \tag{5}$$

$$\psi = \sqrt{(\nu_a\, x\, u_j\, \tilde{c})}\, f(\eta) \tag{6}$$

where $\tilde{c}$ is some constant to be specified later. The modified Chapman–Rubesin description

$$\frac{\rho\mu}{(\rho\mu)_a} = \tilde{c}\lambda(\eta) \tag{7}$$

takes into account the effect of inhomogeneity of the medium, or more simply allows for a separation of the temperature and concentration effects. $\tilde{c}$ is then to be evaluated at some average temperature in the jet, and since one may expect that thermal separation will be small, it is reasonable to assume

$$\lambda(\eta) = 1 \tag{8}$$

so as to decouple the momentum from the remaining relations. With C_1'' substituted into the energy equation the final forms are

$$f''' + ff'' = 0 \tag{9}$$

$$\left.\begin{aligned} & C_1'' + C_1'\left\{Scf + Sc\left(\frac{1}{Sc}\right)' + k_T\,\frac{\bar{T}'}{\bar{T}}\right. \\ & \quad \left.[2(1 - m_{12}) + 2C_1(m_{12} + m_{21} - 2)]\right\} \\ & \quad + Sc\left(\frac{1}{Sc}\right)'\left\{\frac{m_1 m_2}{m^2}\,k_T\,\frac{\bar{T}'}{\bar{T}}\right\} \\ & \quad + \left(k_T\,\frac{\bar{T}}{\bar{T}}\right)'\frac{m_1 m_2}{m^2} = 0 \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} & \frac{C_p}{Pr}\,\bar{T}'' + \bar{T}'\left\{C_p f + \left(\frac{C_p}{Pr}\right)'\right. \\ & \quad + \frac{1}{Sc}\left[C_1(C_{p1} - C_{p2}) + C_1'\,\frac{R\,k_T}{C_1 C_2}\right. \\ & \quad \left.\left. + \left(\frac{Rk_T}{C_1 C_2}\right)'\frac{m_1 m_2}{m^2}\,k_T\right]\right\} \\ & \quad - \frac{R\bar{T}k_T}{C_1 C_2}\,C_1' f + \frac{1}{Sc}\left\{C_1'\left(\frac{Rk_T}{C_1 C_2}\right)'\right. \\ & \quad \bar{T} + \frac{m_1 m_2}{m^2}\,k_T\,\frac{\bar{T}'^2}{\bar{T}}\left[(C_{p1} - C_{p2})\right. \\ & \quad \left.\left. + \frac{Rk_T}{C_1 C_2}\right]\right\} = 0 \end{aligned}\right\} \tag{11}$$

Appropriate boundary conditions are, for the jet core

$$\begin{array}{ccr} \underline{\eta \to -\infty} & \underline{y \to -\infty} & \\ f' = 2 & u = u_j & (12) \\ f = 2\eta & v = 0 & (13) \\ \bar{T} = \bar{T}(-\infty) & \bar{T} = \bar{T}_j & (14) \end{array}$$

and for the dividing streamline

$$\begin{array}{ccr} \underline{\eta = \eta^*} & \underline{y = y^*} & \\ C_1' + \dfrac{m_1 m_2}{m^2}\,k_T\,\dfrac{\bar{T}'}{\bar{T}} = 0 & J^*_{m_1} = 0 & (15) \\ C_1 = C_1(\eta^*) & C_1 = C_1^* & (16) \end{array}$$

and for the ambient gas limit

$$\begin{array}{ccr} \underline{\eta \to \infty} & \underline{y \to +\infty} & \\ f' = 0 & u = 0 & (17) \\ \bar{T} = \bar{T}(+\infty) & \bar{T} = \bar{T}_a & (18) \end{array}$$

Symmetry dictates the conditions (13) that the y component of velocity (v) vanishes at $y \to -\infty$. Equation (15) results from a balance of thermal and concentration gradients on the dividing streamline, i.e. on

$$f(\eta^*) = 0 \tag{19}$$

Due to symmetry, the jet center effectively moves the dividing streamline outward.

Equation (16) is a convenient means for initiating the integration process. Effectively, each $C_1(+\infty)$ is compatible with some $C_1(-\infty)$ [and (12)–(14), (17), (18)] such that (15) is satisfied. In practice, however, $C_1(+\infty)$ need not be specified *a priori* if one is not interested in specific levels *per se*. C_1^* is found to be quite close to the average concentration

$$C_0 = \frac{C(+\infty) + C(-\infty)}{2} \tag{20}$$

The remaining boundary conditions are of the usual kind.

Thermal separation is defined as

$$\Delta C = C(+\infty) - C(-\infty) \tag{21}$$

Necessary transport properties are obtainable from kinetic theory [1, 2]. For example, the diffusion coefficient D_{12} may be evaluated on the basis of a Lennard-Jones model [2, p. 539]. Using curve fitting D_{12} may be approximated (for atmospheric pressure) by

$$D_{12} \simeq 0{\cdot}015 \left(\frac{T}{30{\cdot}58}\right)^{1{\cdot}685} \text{cm}^2/\text{s}$$

The temperature dependence of viscosity for pure N_2 and He are available in references 6 and 7 respectively. A reasonable approximation (within 0·5 per cent) is a quadratic variation of the form

$$\mu = A + BT + CT^2 \tag{22}$$

where A, B and C were evaluated on a least square deviation basis. In detail the mixture viscosity depends upon both temperature and concentration level. In anticipation that thermal separation would not be large, the data was fitted for constant concentration. Within the temperature range of interest $C_p \times 10^{-7} = 5{\cdot}19$, 1·05 ergs/g/°K respectively for helium, nitrogen [8, 9] and after evaluating k in a manner analogous to μ above, the Prandtl number may also be expressed as in equation (22) with appropriate constants.

The Schmidt number does prove to vary appreciably with concentration and was evaluated from the basic definition $Sc = \mu/\rho D_{12}$ with the ideal gas density being

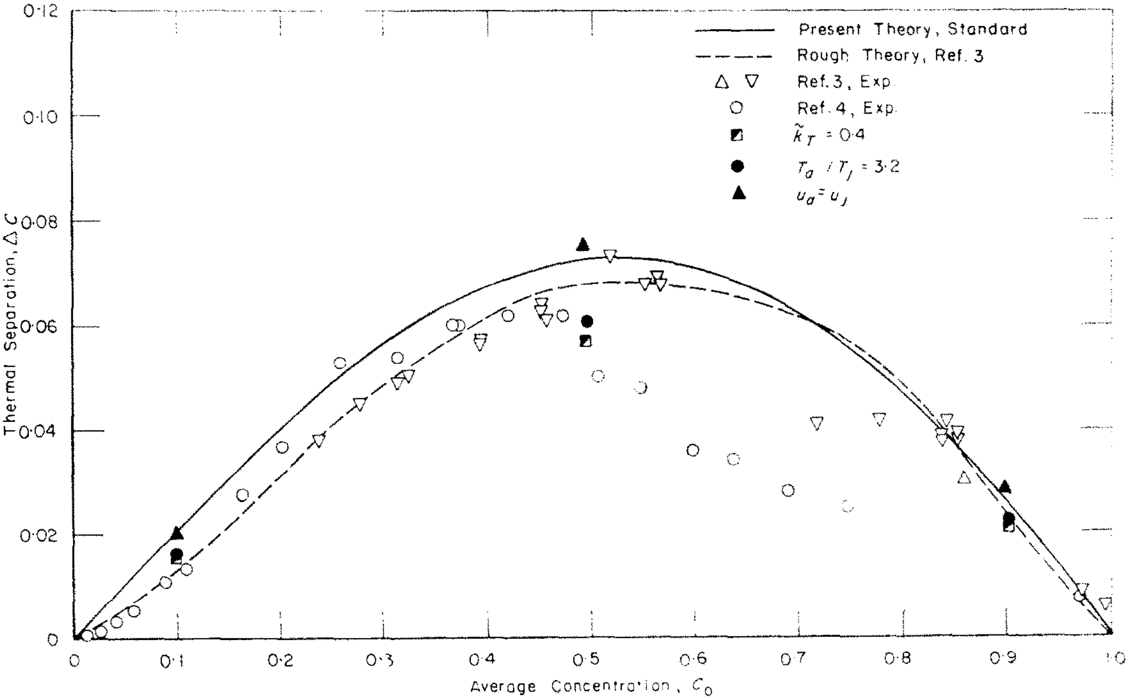

FIG. 2. Effects of thermal diffusion.

$$\rho = \frac{1{\cdot}013}{(2{\cdot}969 + 17{\cdot}811\, C_1)\, T}\ \text{gm/cm}^3$$

Finally, k_T was calculated on the basis of a rigid sphere model [1] but with allowance for a correction factor

$$\tilde{k}_T = \frac{k_T}{(k_T)_{\text{rigid sphere}}} \qquad (23)$$

The rigid sphere expression is

$$(k_T)_{\text{rigid sphere}} = - \frac{12{\cdot}36\, C_1 + 2{\cdot}34}{(1 + 6C_1)\left[11{\cdot}97\dfrac{C_1}{1 - C_1} + 0{\cdot}89\dfrac{1 - C_1}{C_1} + 7{\cdot}63\right]}$$

and $\tilde{k}_T$ generally is considered to be 0·5 [1].

SOLUTION

The decoupled momentum equation may be solved independently using the asymptotic conditions at the jet core ($\eta \rightarrow -\infty$) to initiate the integration process. From equations (9), (12), (13), and (17) it is clear that $f \approx 2\eta + f_2$, $f_2 \ll 2|\eta|$ for sufficiently large negative η. Thus

$$\left.\begin{aligned} f'' &= \beta \exp(-\eta^2) \\ f' &= 2 + \frac{\beta\sqrt{\pi}}{2}\operatorname{erfc}(-\eta) \\ f &= 2\eta + \frac{\beta\sqrt{\pi}}{2}\left\{\eta \operatorname{erfc}(-\eta) \right. \\ &\qquad\qquad \left. + \frac{\exp(-\eta^2)}{\sqrt{\pi}}\right\}. \end{aligned}\right\} \qquad (24)$$

The parameter β effectively replaces condition (17) but must be chosen by some iteration process to be consistent with (17). The dividing streamline is found to be at $\eta^* = 0{\cdot}26455$ with

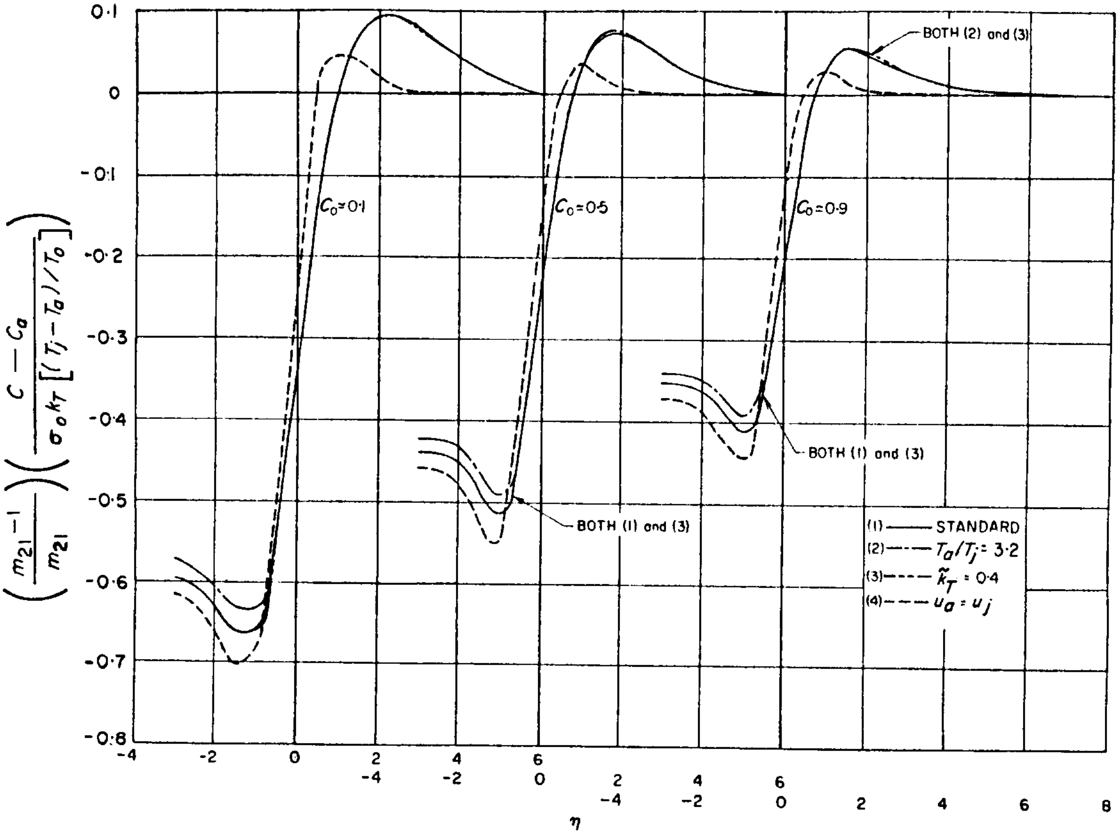

FIG. 3. Profiles for average concentration.

the associated values $f(\eta^*) = 0$, $f'(\eta^*) = 1{\cdot}1742$, and $f''(\eta^*) = -0{\cdot}79907$.

The remaining energy and concentration equations (10), (11) were solved by integrating in both directions from η^*, with assumptions for C_1^*, $\bar{T}^*$ and $\bar{T}'^*$ and thus $C_1'^*$ from (15). Outer boundary conditions were matched after repeated integrations with the aid of a linear influence coefficient scheme as a guide for successive guesses of $\bar{T}^*$ and $\bar{T}'^*$. The results are shown in Figs. 2–3.

A "standard" case is defined for the situation $u_a = 0$, temperature ratio across the shear layer equal to 4·0 and $\tilde{k}_T = 0{\cdot}5$. An evaluation of the importance of such parameters was made by consideration of subcases with *either* no velocity difference, *or* $\bar{T}(\infty)/\bar{T}(-\infty) = 3{\cdot}2$, *or* $\tilde{k}_T = 0{\cdot}4$. For the constant velocity subcase the momentum solution is simply $f = 2\eta$ and $\eta^* = 0$. The shaded points added to Fig. 2 ($C_0 \approx 0{\cdot}1$, 0·5 and 0·9) indicate the velocity ratio effect to be small and the temperature and thermal diffusion ratio correction factor changes to be comparable. Overall, none of these are critical as to the magnitude of the separation effect. Comparison is made with experimental data [3, 4] and an approximate analysis earlier thought to be crude. One also observes in Fig. 2 the relative amplification (by up to an order of magnitude) of uncertainties in mass concentration relative to molar concentration for $C_0 > 0{\cdot}5$. On a mole concentration basis such difference would be virtually absent.

Shear layer profiles are shown in Fig. 3 The parameter $\sigma_0 = [1 + (m_{21} - 1)C_0]^2/m_{21}$ is introduced as a result of the approximate analysis which found that the ordinate (in Fig. 3) should be independent of C_0. Clearly this is not so in detail and σ_0 is not an appropriate scaling parameter. The subcases do not differ markedly in the profile predictions, especially with regard to the primary concentration gradient within the shear layer.

With $\tilde{k}_T = 0{\cdot}5$, the agreement between the present analysis and the initial experiment [3] is reasonably good. The approximate analysis [3] gives surprisingly good overall predictions, but fails to give the correct detail picture. The differences between the two sets of experimental data [3, 4] are much less apparent on a mole concentration display, which was the original data source. Most interesting is verification of maxima and minima in the profiles as was predicted by the approximate analysis [3] although the one profile measurement (Fig. 9 of reference 3) showed only the minimum. Since velocity ratio does not prove to have a dominant influence, the experimental techniques do appear to have some value for dynamic evaluation of the thermal diffusion ratio.

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Résumé—On s'intéresse à la séparation thermique de constituants à travers la couche limite libre séparant un jet froid de mélange gazeux (d'hélium et d'azote) et un gaz chaud ambiant, tous les deux possédant initialement les mêmes degrés de concentration. Une solution asymptotique, en régime permanent, est obtenue en supposant que la couche est peu épaisse par rapport au rayon du jet. A cause de la non-homogénéité peu importante, l'équation de l'impulsion peut être découplée de celle de l'énergie et il en résulte des profils de vitesse connus a priori si l'on suppose comme d'habitude que le produit de la masse volumique par la viscosité est constant. L'intégration par itération des équations de la concentration et de l'énergie est achevée en s'imposant la condition qu'en régime permanent le transport de masse est nul à travers la ligne de courant de séparation et en employant des coordonnées de similitude. Un accord raisonnable avec les données expérimentales disponibles en résulte si l'on emploie le modèle modifié des sphères des sphéres indéformables pour calculer le rapport de diffusion thermique.

Zusammenfassung—Die thermische Trennung von Komponenten an der Reibungsschicht zwischen einem kalten Gemischstrahl (bestehend aus Helium und Stickstoff) und einem heissen umgebenden Gas die beide ursprünglich gleiches Konzentrationsniveau besassen wird untersucht. Eine asymptotische Lösung für den stationären Zustand ergibt sich für die Annahme, dass die Reibungsschicht dünn ist im Vergleich zum Strahlradius. Als eine Folge kleiner Inhomogenitäten kann die Impulsgleichung getrennt werden von Energiebetrachtungen und es ergeben sich a priori bekannte Geschwindigkeitsprofile bei der üblichen Annahme für konstantes Produkt von Dichte und Zähigkeit. Eine schrittweise Integration der Konzentration und Energiebeziehungen ist durchgeführt auf Grund der Bedingung, dass der Stofftransport über die trennende Stromlinie im stationären Zustand Null ist und unter Benützung von Ähnlichkeitskoordinaten. Zufriedenstellende Übereinstimmung mit verfügbaren Versuchsdaten ergibt sich bei Benützung des modifizierten Festkugelmodells für das thermische Diffusionsverhältnis.

Аннотация—Рассмотрено терморазделение компонентов поперек пограничного слоя между холодной струей (из смеси гелия и азота) и горячим окружающим воздухом, причем концентрация обоих вначале одинакова. При допущении, что толщина слоя мала по сравнению с радиусом струи, получено асимптотическое стационарное решение. В результате наличия небольшой неоднородности уравнение количества движения можно разделить, исходя из соображений энергии и известных априори профилей скорости при обычном допущении постоянства произведения плотности и вязкости. На основе стационарного условия отсутствия переноса массы через разделяющую линию тока проведено итеративное интегрирование соотношения концентрации и энергии, используя автомодельные переменные. Получено приемлемое согласование с имеющимися экспериментальными данными для термодиффузионного отношения при использовании модифицированной модели твердой сферы.